Modeling Error of α-Models of Turbulence on a Two-Dimensional Torus

Luigi C. Berselli[1], Argus Adrian Dunca[2], Roger Lewandowski[3], and Dinh Duong Nguyen[4]

[1] Università di Pisa, Dipartimento di Matematica, Via Buonarroti 1/c, I-56127 Pisa, Italy, E-mail: luigi.carlo.berselli@unipi.it
[2]Department of Mathematics and Informatics, University Politehnica of Bucharest, Bucharest, Romania, E-mail: argus_dunca@yahoo.com
[3,4]IRMAR, UMR CNRS 6625, University of Rennes 1 and FLUMINANCE Team, INRIA Rennes, France , E-mail: Roger.Lewandowski@univ-rennes1.fr, dinh-duong.nguyen@univ-rennes1.fr

Abstract

This paper is devoted to study the rate of convergence of the weak solutions $\mathbf{u}_\alpha$ of α-regularization models to the weak solution $\mathbf{u}$ of the Navier-Stokes equations in the two-dimensional periodic case, as the regularization parameter α goes to zero. More specifically, we will consider the Leray-α, the simplified Bardina, and the modified Leray-α models. Our aim is to improve known results in terms of convergence rates and also to show estimates valid over long time intervals.

Key words : Rate of convergence, α-turbulence models, Navier-Stokes Equations.
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1 Introduction

In this work we study the rates of convergence of weak solutions of several two dimensional α-models of turbulence to the weak solution of the Navier-Stokes equations (NSE), with periodic boundary conditions. We work mainly in two space dimensions, even if some remarks concerning the three dimensional case are given in Section 6. The turbulence models we study belong to the class of Large Eddy Simulation models (LES), used to carry out numerical simulations of turbulence flows, that cannot be performed by the NSE because. In fact, according to Kolmogorov laws, it would require $\mathcal{O}(Re^{d^2/4})$ degrees of freedom where $d = 2, 3$, which is still inaccessible to modern computers, for higher (real-life) Reynolds numbers [2, 8]. The motivation to consider the 2D case is because this setting is appropriate to analyse models that simulate layers of shallow water in stratified flows, such as those occurring in the ocean or in the atmosphere [7, 29].
Let $L > 0$ denote a given length scale, $\mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x})$ for $t > 0$ and $\mathbf{x} \in \mathbb{R}^2/[0, L]^2 = \mathbb{T}_2$, denote the velocity and the pressure of and incompressible fluid, which satisfies the NSE,

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } (0, T) \times \mathbb{T}_2, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } (0, T) \times \mathbb{T}_2, \tag{1.2}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \qquad \text{in } \mathbb{T}_2, \tag{1.3}$$

where the constant $\nu > 0$ denotes the kinematic viscosity, $\mathbf{u}_0$ and $\mathbf{f}$ are given as the initial velocity and the external force. The α-models aim at regularizing the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$, and are are given by the following general abstract form

$$\partial_t \mathbf{u}_\alpha + N(\mathbf{u}_\alpha) - \nu\Delta\mathbf{u}_\alpha + \nabla p_\alpha = \mathbf{f}, \qquad \text{in } (0,T)\times\mathbb{T}_2, \tag{1.4}$$

$$\nabla \cdot \mathbf{u}_\alpha = 0, \qquad \text{in } (0,T)\times\mathbb{T}_2, \tag{1.5}$$

$$\mathbf{u}_\alpha|_{t=0} = \mathbf{u}_0 \qquad \text{in } \mathbb{T}_2, \tag{1.6}$$

where, for $\alpha > 0$ the fields $\mathbf{u}_\alpha$ and p_α are the filtered velocity and pressure, respectively, at frequencies of order $1/\alpha$. The α-models under study herein are: the Leray-α, the simplified Bardina, and the modified Leray-α models, given by

$$N(\mathbf{u}_\alpha) = \begin{cases} (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha & \text{Leray-}\alpha\text{ (L-}\alpha\text{)}, \\ (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha & \text{Bardina model (SB)}, \\ (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha & \text{Modified Leray-}\alpha\text{ (ML-}\alpha\text{)}, \end{cases} \tag{1.7}$$

and the bar operator is given by solving the Helmholtz equation

$$\overline{\mathbf{v}} - \alpha^2\Delta\overline{\mathbf{v}} = \mathbf{v} \quad \text{in } \mathbb{T}_2, \tag{1.8}$$

in the setting of periodic functions with zero mean value.
The first model $N_\alpha(\mathbf{v}) = (\overline{\mathbf{v}} \cdot \nabla)\mathbf{v}$ is due to J. Leray [27], who considered the problem in the whole space $\mathbb{R}^3$, and where $\overline{\mathbf{v}} = \mathbf{v} * \rho_\alpha$, for a standard mollifier ρ_α. Note that in the whole space it is also possible to explicitly write a Kernel G_α such that $\overline{\mathbf{v}} = \mathbf{v} * G_\alpha$ for the Helmholtz filter [30]. The class of α-models has been the subject of many investigations in the last two decades, see for instance [5, 11, 12, 10, 13, 16, 17, 21, 22, 24, 25]. It is known that the Cauchy problem has global, unique, and regular solutions, with $\mathbf{u}_\alpha$ at least in $L^\infty_t H^1_x \cap L^2_t H^2_x$. These solutions converge to solutions of the NSE as $\alpha \to 0$. which means that $\mathbf{u}_\alpha \to \mathbf{u}$, $p_\alpha \to p$, where $(\mathbf{u}, p)$ is the corresponding weak solution of the NSE, under suitable assumptions about the data,
In this paper we will study the rate of convergence as $\alpha \to 0$, namely the norm of

$$\mathbf{e} := \mathbf{u} - \mathbf{u}_\alpha, \tag{1.9}$$

in various space such as $L^\infty_t L^2_x$, $L^2_t H^1_x$, $L^\infty_t H^1_x$ and $L^2_t H^2_x$.
This study is motivated by the results in Cao and Titi [6], in which the authors proved that for all 2D α-models (1.4)-(1.7), the following $L^\infty_t L^2_x$ estimate holds true on a given time interval $[0,T]$

$$\sup_{t\in[0,T]} \|\overline{\mathbf{u}}_\alpha(t) - \mathbf{u}(t)\|^2 \le C\alpha^2 \left(CT \left(1 + \log\left(\frac{L}{2\pi\alpha}\right)\right) + C\right) \qquad \forall\, \alpha \le \frac{L}{2\pi}, \tag{1.10}$$

where C is a constant and when no risk of confusion occurs, $\|\cdot\|$ stands for the L^2-norm. To prove the convergence rate (1.10) it is assumed that

$$\mathbf{u}_0 \in \mathcal{D}(-P_\sigma\Delta) \qquad \text{and} \qquad \mathbf{f} \in L^2([0,T], \mathcal{P}_\sigma L^2(\mathbb{T}_2)^2),$$

P_σ being the Leray projector. The logarithmic factor that appears in (1.10) comes from the application of an inequality initially proved by Brézis and Gallouët in [4].
Cao-Titi's result raises two questions:

i) Is it possible to improve the $O(\alpha^2 \log(1/\alpha))$ rate, and what about the convergence rate in stronger norms?

ii) Is it possible to prove an estimate global in time?

In this paper we positively answer to both these questions by showing that when

$$\mathbf{u}_0 \in \mathcal{P}_\sigma H^1(\mathbb{T}_2)^2 \qquad \text{and} \qquad \mathbf{f} \in L^2(\mathbb{R}_+; \mathcal{P}_\sigma L^2(\mathbb{T}_2)^2),$$

we get an estimate uniform in time of order $O(\alpha^3)$ in the $L^\infty_t L^2_x \cap L^2_t H^1_x$ norms. More specifically we prove that for all α-models (1.4)-(1.7), it holds

$$\|\mathbf{e}(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{e}\|^2 \, dt \leq C\alpha^3 \qquad \forall\, s \geq 0, \tag{1.11}$$

where C is a time-independent constant, see Theorem 4.1 below. We also get a uniform in time estimate in the $L^\infty_t H^1_x \cap L^2_t H^2_x$ norm of order $O(\alpha^2)$ for the L-α model, and in $O(\alpha^2 \log(1/\alpha))$ for SB and ML-α model, namely for all $s \geq 0$, we will prove

$$\|\nabla \mathbf{e}(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{e}\|^2 \, dt \leq \begin{cases} C\alpha^2 & \text{for L-}\alpha, \\ C\alpha^2 \left(C \log\left(\dfrac{L}{2\pi\alpha}\right) + C \right) & \text{for SB and ML-}\alpha. \end{cases} \tag{1.12}$$

see Theorem 4.2 below. Estimates (1.11) and (1.12) are the main results in the present work.
Thanks to (1.11)-(1.12), we are also able to study the rates of convergence of the pressures, by showing (see Theorem 5.1 below) that

$$\int_0^s \|\nabla q\|^2 \, dt \leq \begin{cases} C\alpha^{5/2} & \text{for L-}\alpha\ , \\ C\alpha^2 \left(C \log\left(\dfrac{L}{2\pi\alpha}\right) + C \right) & \text{for SB and ML-}\alpha, \end{cases}$$

where C is independent of the time and $q = p_\alpha - p$.

Plan of the paper: The paper is organized as follows. In Section 2 we set the mathematical framework. In Section 3 we derive from energy balances uniform-in-time energy(type) estimates for weak solutions of the NSE and for all α-models as well. This is the main step, before investigating the rates of convergence in Section 4, where we prove the estimates (1.11)-(1.12). Section 5 is devoted to study of the convergence rate for the pressure. In Section 6, we make some additional remarks about the 3D case for which the situation is quite different and not in the focus of the present paper.

2 Mathematical framework

In this section we set the functional spaces we are working with. We show basic properties of the Helmholtz filter, then we carry out the Leray projection of the NSE and Leray-α on divergence-free fields spaces. The section ends with a brief state of the art about α-models.

2.1 Function spaces

Let $\mathbb{T}_2 := \mathbb{R}^2/[0, L]^2$ be a 2D torus; for $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, let $L^p(\mathbb{T}_2)$ and $H^m(\mathbb{T}_2)$ denote the standard Lebesgue and Sobolev spaces on $\mathbb{T}_2$, respectively. The $L^p(\mathbb{T}_2)$-norm is denoted by $\| \cdot \|_p$ for all $1 \leq p \leq \infty$, except for the case $p = 2$ where $\| \cdot \| \equiv \| \cdot \|_2$.

Boldface symbols are used for vectors, matrices, or space of vectors. We denote by Π the set of all trigonometric polynomials on $\mathbb{T}_2$ with spatial zero mean, *i.e.*,

$$\int_{\mathbb{T}_2} \phi(\mathbf{x})\, d\mathbf{x} = 0, \quad \forall\, \phi \in \Pi.$$

Let us define

$$\boldsymbol{\Lambda} := \left\{ \boldsymbol{\varphi} \in \Pi^2 \,:\, \nabla \cdot \boldsymbol{\varphi} = 0 \right\}.$$

As usual when studying the NSE we define the following standard Hilbert functional spaces

$$\begin{aligned} \mathbf{H} &:= \text{the closure of } \boldsymbol{\Lambda} \text{ in } L^2(\mathbb{T}_2)^2, \\ \mathbf{V} &:= \text{the closure of } \boldsymbol{\Lambda} \text{ in } H^1(\mathbb{T}_2)^2. \end{aligned}$$

Let $(\cdot,\cdot)$ and $\|\cdot\|$ be the standard inner product and norm on $\mathbf{H}$, that are

$$(\mathbf{u},\mathbf{v}) := \int_{\mathbb{T}_2} \mathbf{u}\cdot\mathbf{v}\, d\mathbf{x} \quad \text{and} \quad \|\mathbf{u}\|^2 := \int_{\mathbb{T}_2} |\mathbf{u}|^2\, d\mathbf{x}.$$

The inner product $(\mathbf{u},\mathbf{v})_{\mathbf{V}}$ and the corresponding norm $\|\mathbf{u}\|_{\mathbf{V}}$ on $\mathbf{V}$ are defined as follows

$$(\mathbf{u},\mathbf{v})_{\mathbf{V}} := (\nabla\mathbf{u}, \nabla\mathbf{v}) \quad \text{and} \quad \|\mathbf{u}\|_{\mathbf{V}} := \|\nabla\mathbf{u}\|.$$

In the sequel, we use the symbol P_σ to denote the Helmholtz-Leray orthogonal projection operator of $\mathbf{L}^2(\mathbb{T}_2)$ onto $\mathbf{H}$. We next consider an orthonormal basis $\{\boldsymbol{\varphi}_j\}_{j\in\mathbb{N}}$, of $\mathbf{H}$ consisting of eigenfunctions of the Laplace operator

$$-\Delta \,:\, \mathbf{H}^2(\mathbb{T}_2) \cap \mathbf{V} \longrightarrow \mathbf{H},$$

and for $m \geq 1$, $\mathbf{H}_m := \operatorname{span}\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_m\}$.

Let $A = -P_\sigma\Delta$ be the Stokes operator, with domain $\mathcal{D}(A) := \mathbf{H}^2(\mathbb{T}_2) \cap \mathbf{V}$. Then, it is well-known (cf. [6, 17]) that:

$$A\mathbf{u} = -P_\sigma\Delta\mathbf{u} = -\Delta\mathbf{u} \qquad \forall\, \mathbf{u} \in \mathcal{D}(A).$$

Let $\lambda_1 > 0$ be the first eigenvalue of A, i.e., $A\boldsymbol{\varphi}_1 = \lambda_1\boldsymbol{\varphi}_1$, and the above setting leads to $\lambda_1 = (2\pi/L)^2$. By virtue of the Poincaré inequality we have

$$\lambda_1\|\mathbf{u}\|^2 \leq \|\nabla\mathbf{u}\|^2 \qquad \forall\, \mathbf{u} \in V, \tag{2.1}$$

$$\lambda_1\|\nabla\mathbf{u}\|^2 \leq \|A\mathbf{u}\|^2 = \|\Delta\mathbf{u}\|^2 \qquad \forall\, \mathbf{u} \in \mathcal{D}(A). \tag{2.2}$$

Then, it follows by (2.1)-(2.2) that there exist positive dimensionless constants c_1, c_2 such that

$$c_1\|A\mathbf{u}\| \leq \|\mathbf{u}\|_{\mathbf{H}^2(\mathbb{T}_2)} \leq c_2\|A\mathbf{u}\| \qquad \forall\, \mathbf{u} \in \mathcal{D}(A).$$

In the following, we will make an intensive use of the 2D-Ladyženskaya inequality [23]:

$$\|\mathbf{u}\|_4 \leq C\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2} \qquad \forall\, \mathbf{u} \in \mathbf{V}, \tag{2.3}$$

where C is a non-negative dimensionless constant.

2.2 On the Helmholtz filter

The filter operator used to construct the turbulence models is the differential filter associated with the Helmholtz filter, see Germano [19], or [3, 15, 26]. Given a cut length $\alpha > 0$ (which will be called the filter radius), for each $\mathbf{u} \in \mathbf{H}$, then $\overline{\mathbf{u}} \in \mathbf{H}^2(\mathbb{T}_2) \cap \mathbf{V}$ is the unique solution of the following Helmholtz equation (1.8). By a direct calculation, from (1.8) we deduce

$$\|\mathbf{u} - \overline{\mathbf{u}}\| = \alpha^2 \|\Delta \overline{\mathbf{u}}\| \qquad \forall\, \mathbf{u} \in \mathbf{H}.$$

Moreover, we already know that the filter satisfies the following inequality, see [14]:

$$\|\overline{\mathbf{u}}\| + \alpha \|\nabla \overline{\mathbf{u}}\| + \alpha^2 \|\Delta \overline{\mathbf{u}}\| \le C \|\mathbf{u}\| \qquad \forall\, \mathbf{u} \in \mathbf{H}, \tag{2.4}$$

where C is a Sobolev constant. It follows that

$$\|\nabla \mathbf{u} - \nabla \overline{\mathbf{u}}\| = \alpha^2 \|\nabla \Delta \overline{\mathbf{u}}\| \le C\alpha \|\Delta \mathbf{u}\| \qquad \forall\, \mathbf{u} \in \mathcal{D}(A). \tag{2.5}$$

2.3 On the Leray projection operator

Throughout the rest of the paper we assume

$$\mathbf{u}_0 \in \mathbf{V} \qquad \text{and} \qquad \mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H}). \tag{2.6}$$

In order to eliminate the pressure from the equations, we apply the Helmholtz-Leray orthogonal projection $\mathcal{P}_\sigma : L^2(\mathbb{T}_2)^2 \to \mathbf{H}$ on divergence-free fields to both the NSE and α-models. We get the following functional equations:

$$\begin{aligned} \frac{d\mathbf{u}}{dt} + P_\sigma[(\mathbf{u} \cdot \nabla)\mathbf{u}] - \nu \Delta \mathbf{u} &= \mathbf{f}, \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0, \end{aligned} \tag{2.7}$$

as well as

$$\begin{aligned} \frac{d\mathbf{u}_\alpha}{dt} + P_\sigma[N(\mathbf{u}_\alpha)] - \nu \Delta \mathbf{u}_\alpha &= \mathbf{f}, \\ \mathbf{u}_\alpha|_{t=0} &= \mathbf{u}_0, \end{aligned} \tag{2.8}$$

where we used the facts that $P_\sigma \mathbf{f} \equiv \mathbf{f}$ since $\mathbf{f} \in \mathbf{H}$, $P_\sigma \Delta \mathbf{u} = \Delta \mathbf{u}$ due to the periodic setting, and $P_\sigma(\nabla p) = P_\sigma(\nabla p_\alpha) = 0$. Once the velocity is calculated, the pressures p and p_α are solutions of the following Poisson equations

$$-\Delta p = \nabla \cdot ((\mathbf{u} \cdot \nabla)\mathbf{u}) \qquad \text{and} \qquad -\Delta p_\alpha = \nabla \cdot (N(\mathbf{u}_\alpha)).$$

From now when speaking of solutions to the NSE and to α-models will only consider the velocities, and the pressure can be associated by solving the above equations.

Remark 2.1. *Thanks to the Leray-Helmholtz decomposition and for simplicity we assume that* $\mathbf{f}$ *is divergence free. If not the case, the gradient part of* $\mathbf{f}$ *can be added to the pressure (to obtain a modified pressure) and* $P_\sigma \mathbf{f}$ *will replace* $\mathbf{f}$*.*

Remark 2.2. *A common property of all* α*-models considered in the present paper is that these models "formally" reduce to the NSE when* $\alpha = 0$*. It can be seen directly from the equality (1.8).*

2.4 Brief State of the art

It is well-known that in the 2D case, there exists a unique solution of the NSE, global in time, without formation of singularities, see Temam [34, 35]. Nevertheless, this does not resolve the computational issues of the shallow waters or of stratified flows.
The proof of the existence and uniqueness of solution of the α-models given by (1.7) can be established by using the standard Galerkin method. The L-α model was implemented computationally by Cheskidov-Holm-Olson-Titi [13]. Ilyin-Lunasin-Titi introduced and studied the ML-α model in the 3D periodic case, see [22] and it was tested numerically in [20]. However, the global existence and uniqueness for 2D can be proved in a similar way.
The Bardina closure model of turbulence was first introduced by Bardina-Ferziger-Reynolds in [1] to perform simulations of the atmosphere. A simplified version of the Bardina's model, was modeled and studied in [24, 25], then in [30] the whole space case was studied. This model is designed by $N(\mathbf{u}_\alpha) = \overline{\nabla \cdot (\mathbf{u}_\alpha \otimes \mathbf{u}_\alpha)}$. Cao-Lunasin-Titi proposed a variant of this model [5], which is the one we consider in this paper and that we still call "Simplified Bardina model" (SB).

3 A priori estimates

3.1 General Orientation

As the data are as in (2.6) it is well-known that both the NSE (1.1)-(1.3) and the α-model (1.5)-(1.6) (for any nonlinearity $N(\mathbf{u}_\alpha)$ as those given in (1.7)) admit a unique solution $\mathbf{u}$ such that

$$\mathbf{u} \in L^\infty(\mathbb{R}_+; \mathbf{V}) \cap L^2(\mathbb{R}_+; \mathbf{H}^2(\mathbb{T}_2) \cap \mathbf{V}).$$

To shorten the notation in the following we set

$$\mathcal{F} := \|\mathbf{f}\|^2_{\mathbf{L}^2(\mathbb{R}_+;\mathbf{H})}.$$

In this section, we detail the $L^2(\mathbb{R}_+; \mathbf{H}^2(\mathbb{T}_2) \cap \mathbf{V})$ estimates to get precise constants, for the various models. The analysis is based on 2D energy inequalities, using the Ladyžhenskaya inequality (2.3). About the α-models, we start by estimating $\overline{\mathbf{u}}_\alpha$. This analysis is based on the following identities.

$$(\mathcal{P}_\sigma((\mathbf{u}\cdot\nabla)\mathbf{u}), \mathbf{u}) = (\mathcal{P}_\sigma((\mathbf{u}\cdot\nabla)\mathbf{u}), \Delta\mathbf{u}) = 0 \qquad \forall\, \mathbf{u} \in \mathcal{D}(A), \tag{3.1}$$

which are well-known [34] and on the extension to the L-α and the SB,

$$(\mathcal{P}_\sigma N(\mathbf{u}_\alpha), \mathbf{u}_\alpha) = (\mathcal{P}_\sigma N(\mathbf{u}_\alpha), \Delta\overline{\mathbf{u}}_\alpha) = 0 \qquad \forall\, \mathbf{u}_\alpha \in \mathbf{V}.$$

However, the nonlinearity in the ML-α model is less favorable, since we only have

$$(\mathcal{P}_\sigma N(\mathbf{u}_\alpha), \overline{\mathbf{u}}_\alpha) = 0.$$

3.2 Estimates for the NSE

We recall the basic estimate for weak solutions to the two dimensional NSE.

Lemma 3.1 (**NSE**)**.** *Let* $\mathbf{u}_0 \in \mathbf{V}$ *and* $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$*. Then, the unique weak solution* $\mathbf{u}$ *of the NSE satisfies*

$$\|\mathbf{u}(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{u}\|^2\,dt \le \|\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu\lambda_1} =: C_{NSE1} \qquad \forall\, s \ge 0, \tag{3.2}$$

and

$$\|\nabla \mathbf{u}(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}\|^2\,dt \le \|\nabla \mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu} =: C_{NSE2} \qquad \forall\, s \ge 0. \tag{3.3}$$

Remark 3.1. *Estimate (3.2) in the previous theorem can be obtained more generally when* $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{V}')$*. We use the condition* $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$ *for both estimates (3.2) and (3.3) for shortness.*

Proof. Proofs are well-known but we reproduce them to keep precise track of the constants and to see differences with the other models. We argue step by step, first proving (3.2).
Step 1. $L^2H^1_x \cap L^\infty_t L^2_x$ *estimate.* Take the scalar product of the NSE (2.7) with $\mathbf{u}$ and use the identity $(P_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}],\mathbf{u}) = 0$, which lead to the following estimate

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \nu\|\nabla \mathbf{u}\|^2 \le \|\mathbf{f}\|\|\mathbf{u}\|. \tag{3.4}$$

Using Poincaré and Young inequalities on the r.h.s (right-hand side) of (3.4) yields:

$$\frac{d}{dt}\|\mathbf{u}\|^2 + \nu\|\nabla \mathbf{u}\|^2 \le \frac{1}{\nu\lambda_1}\|\mathbf{f}\|^2. \tag{3.5}$$

Integrating (3.5) on $[0,s]$ for $s \ge 0$, one has

$$\|\mathbf{u}(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{u}\|^2\,dt \le \|\mathbf{u}_0\|^2 + \frac{1}{\nu\lambda_1}\int_0^s \|\mathbf{f}\|^2\,dt. \tag{3.6}$$

Finally, the estimate (3.2) follows by (3.6) since $s \ge 0$ can be chosen arbitrary.
Step 2. $L^2H^2_x \cap L^\infty_t H^1_x$ *estimate.* In order to prove the estimate (3.3), we take $-\Delta\mathbf{u}$ as a test function for the NSE (2.7). As we already have said, in the 2D case periodic the nonlinear term vanishes, cf. (3.1). By the Young inequality the term corresponding to the body force can be estimated by

$$(\mathbf{f}, -\Delta\mathbf{u}) \le \frac{1}{2\nu}\|\mathbf{f}\|^2 + \frac{\nu}{2}\|\Delta\mathbf{u}\|^2,$$

and the rest of the proof follows as for the first estimate. Thus, the proof is complete. □

3.3 Estimates for the Leray-α model

We now prove a uniform estimate for weak solutions to the Leray-α model.

Lemma 3.2 (**L-**α)**.** *Let* $\mathbf{u}_0 \in \mathbf{V}$ *and* $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$*. Then, the unique weak solution* $\mathbf{u}_\alpha$ *of the L-*α *satisfies the following energy-type estimate.*

$$\|\nabla \mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2\,dt \le \frac{CC_{L1}^2}{\nu^4}\left(\|\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu\lambda_1}\right) + \frac{2\mathcal{F}}{\nu} =: C_L \qquad \forall\, s \ge 0, \tag{3.7}$$

where C_{L1} *is given in (3.11).*

Proof. For the L-α model, we recall the nonlinear term is given by

$$N(\mathbf{u}_\alpha) = (\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha \quad \text{where} \quad \bar{\mathbf{u}}_\alpha - \alpha^2 \Delta \bar{\mathbf{u}}_\alpha = \mathbf{u}_\alpha.$$

We argue in 3 steps, with an intermediate step to estimate $\bar{\mathbf{u}}_\alpha$ uniformly in time.

Step 1. $L^2H^1_x \cap L^\infty_t L^2_x$ estimate of $\mathbf{u}_\alpha$. Taking $\mathbf{u}_\alpha$ as a test function in the L-α model (2.8) gives

$$\frac{d}{dt}\|\mathbf{u}_\alpha\|^2 + \nu\|\nabla \mathbf{u}_\alpha\|^2 \le \frac{1}{\nu\lambda_1}\|\mathbf{f}\|^2.$$

Since $(P_\sigma[(\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha], \mathbf{u}_\alpha) = 0$, see [6], this leads to

$$\|\mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{u}_\alpha\|^2 \, dt \le \|\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu\lambda_1} \qquad \forall\, s \ge 0. \tag{3.8}$$

Step 2. $L^2H^2_x \cap L^\infty_t H^1_x$ estimate of $\bar{\mathbf{u}}_\alpha$. Testing (2.8) by $-\Delta\bar{\mathbf{u}}_\alpha$ and replacing $\mathbf{u}_\alpha$ by $\bar{\mathbf{u}}_\alpha - \alpha^2\Delta\bar{\mathbf{u}}_\alpha$ yield

$$\frac{d}{dt}\left(\|\nabla\bar{\mathbf{u}}_\alpha\|^2 + \alpha^2\|\Delta\bar{\mathbf{u}}_\alpha\|^2\right) + \nu\|\Delta\bar{\mathbf{u}}_\alpha\|^2 + 2\nu\alpha^2\|\nabla\Delta\bar{\mathbf{u}}_\alpha\|^2 \le \frac{\|\mathbf{f}\|^2}{\nu}. \tag{3.9}$$

Here, the vanishing of the nonlinear term has been used, i.e.,

$$(P_\sigma[(\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha], -\Delta\bar{\mathbf{u}}_\alpha) = ((\bar{\mathbf{u}}_\alpha \cdot \nabla)(\bar{\mathbf{u}}_\alpha - \alpha^2\Delta\bar{\mathbf{u}}_\alpha), -\Delta\bar{\mathbf{u}}_\alpha) = 0,$$

which is a consequence of (3.1). Therefore, by (3.9) for all $s \ge 0$

$$\|\nabla\bar{\mathbf{u}}_\alpha(s)\|^2 + \alpha^2\|\Delta\bar{\mathbf{u}}_\alpha(s)\|^2 + \nu\int_0^s \left(\|\Delta\bar{\mathbf{u}}_\alpha\|^2 + 2\alpha^2\|\nabla\Delta\bar{\mathbf{u}}_\alpha\|^2\right) dt \le C_{L1}, \tag{3.10}$$

where C_{L1} is given by

$$\|\nabla\bar{\mathbf{u}}_0\|^2 + \alpha^2\|\Delta\bar{\mathbf{u}}_0\|^2 + \frac{\mathcal{F}}{\nu} \le (1+\lambda_1)\|\nabla\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu} =: C_{L1}, \tag{3.11}$$

here the inequalities

$$\|\nabla\bar{\mathbf{u}}_0\| \le \|\nabla\mathbf{u}_0\| \qquad \text{and} \qquad \alpha^2\|\Delta\bar{\mathbf{u}}_0\|^2 \le \|\mathbf{u}_0\|^2,$$

given by (2.4) and the Poincaré inequality have been applied.

Step 3. $L^2H^2_x \cap L^\infty_t H^1_x$ estimate of $\mathbf{u}_\alpha$. We test (2.8) by $-\Delta\mathbf{u}_\alpha$ which leads now to the following equality

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}_\alpha\|^2 + \nu\|\Delta\mathbf{u}_\alpha\|^2 = (P_\sigma[(\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha], \Delta\mathbf{u}_\alpha) + (\mathbf{f}, -\Delta\mathbf{u}_\alpha). \tag{3.12}$$

The first term on the r.h.s of (3.12) can be estimated by:

$$\begin{aligned}(P_\sigma[(\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha], \Delta\mathbf{u}_\alpha) &\le C\|\bar{\mathbf{u}}_\alpha\|_4\|\nabla\mathbf{u}_\alpha\|_4\|\Delta\mathbf{u}_\alpha\| \\ &\le C\|\nabla\bar{\mathbf{u}}_\alpha\|\|\nabla\mathbf{u}_\alpha\|^{1/2}\|\Delta\mathbf{u}_\alpha\|^{3/2} \\ &\le \frac{C}{\nu^3}\|\nabla\bar{\mathbf{u}}_\alpha\|^4\|\nabla\mathbf{u}_\alpha\|^2 + \frac{\nu}{4}\|\Delta\mathbf{u}_\alpha\|^2.\end{aligned} \tag{3.13}$$

Here we used the Hölder, 2D-Ladyžhenskaya (2.3), Sobolev, and Young inequalities, respectively. From (3.12)-(3.13) one obtains

$$\frac{d}{dt}\|\nabla\mathbf{u}_\alpha\|^2 + \nu\|\Delta\mathbf{u}_\alpha\|^2 \le \frac{2}{\nu}\|\mathbf{f}\|^2 + \frac{C}{\nu^3}\|\nabla\bar{\mathbf{u}}_\alpha\|^4\|\nabla\mathbf{u}_\alpha\|^2,$$

which yields

$$\|\nabla\mathbf{u}_\alpha(s)\|^2 + \nu\int_0^s \|\Delta\mathbf{u}_\alpha\|^2\, dt \le \frac{2\mathcal{F}}{\nu} + \frac{C}{\nu^3}\int_0^s \|\nabla\bar{\mathbf{u}}_\alpha\|^4\|\nabla\mathbf{u}_\alpha\|^2\, dt \qquad \forall\, s \ge 0. \tag{3.14}$$

Finally, both estimates (3.8) and (3.10) are applied in (3.14) to get (3.7), which ends the proof. ☐

3.4 Estimates for the simplified Bardina model

In this section we prove a uniform estimate for weak solutions to the simplified Bardina model.

Lemma 3.3 (SB). *Let $\mathbf{u}_0 \in \mathbf{V}$ and let $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$. Then, the unique weak solution $\mathbf{u}_\alpha$ of the SB model satisfies*

$$\|\nabla \mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 \, dt \leq \frac{C C_S^2}{\nu^2 \lambda_1} + \frac{2\mathcal{F}}{\nu} =: C_{SB} \qquad \forall\, s \geq 0,$$

where C is a positive constant and C_S is given by (3.15).

Proof. We recall that for this model, the nonlinear term is given by

$$N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha,$$

and we will prove the global-in-time estimate in two steps, starting with bounds on $\overline{\mathbf{u}}_\alpha$.

Step 3. $L^2 H^2_x \cap L^\infty_t H^1_x$ estimate of $\overline{\mathbf{u}}_\alpha$. Taking $-\Delta \overline{\mathbf{u}}_\alpha$ as a test function in (2.8) and using the fact $\mathbf{u}_\alpha = \overline{\mathbf{u}}_\alpha - \alpha^2 \Delta \overline{\mathbf{u}}_\alpha$ give us

$$\frac{d}{dt}\left(\|\nabla \overline{\mathbf{u}}_\alpha\|^2 + \alpha^2 \|\Delta \overline{\mathbf{u}}_\alpha\|^2\right) + \nu \|\Delta \overline{\mathbf{u}}_\alpha\|^2 + 2\nu\alpha^2 \|\nabla \Delta \overline{\mathbf{u}}_\alpha\|^2 \leq \frac{1}{\nu}\|\mathbf{f}\|^2,$$

where the identity $(P_\sigma[(\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], -\Delta \overline{\mathbf{u}}_\alpha) = 0$, has been used. Thus, we get

$$\|\nabla \overline{\mathbf{u}}_\alpha(s)\|^2 + \alpha^2 \|\Delta \overline{\mathbf{u}}_\alpha(s)\|^2 + \nu \int_0^s \left(\|\Delta \overline{\mathbf{u}}_\alpha\|^2 + 2\alpha^2 \|\nabla \Delta \overline{\mathbf{u}}_\alpha\|^2\right) dt \leq C_S \qquad \forall\, s \geq 0, \tag{3.15}$$

where $C_S := C_{L1}$ as given in (3.11).

Step 2. $L^2 H^2_x \cap L^\infty_t H^1_x$ estimate of $\mathbf{u}_\alpha$. Taking $-\Delta \mathbf{u}_\alpha$ as test function in (2.8) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}_\alpha\|^2 + \nu \|\Delta \mathbf{u}_\alpha\|^2 = (P_\sigma[(\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], \Delta \mathbf{u}_\alpha) - (\mathbf{f}, \Delta \mathbf{u}_\alpha). \tag{3.16}$$

The nonlinear term on the r.h.s of (3.16) is estimated by:

$$\begin{aligned}(P_\sigma[(\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], \Delta \mathbf{u}_\alpha) &\leq C \|\overline{\mathbf{u}}_\alpha\|_4 \|\nabla \overline{\mathbf{u}}_\alpha\|_4 \|\Delta \mathbf{u}_\alpha\| \\ &\leq C \|\overline{\mathbf{u}}_\alpha\|^{1/2} \|\nabla \overline{\mathbf{u}}_\alpha\| \|\Delta \overline{\mathbf{u}}_\alpha\|^{1/2} \|\Delta \mathbf{u}_\alpha\| \\ &\leq \frac{C}{\nu} \|\overline{\mathbf{u}}_\alpha\| \|\nabla \overline{\mathbf{u}}_\alpha\|^2 \|\Delta \overline{\mathbf{u}}_\alpha\| + \frac{\nu}{4}\|\Delta \mathbf{u}_\alpha\|^2.\end{aligned} \tag{3.17}$$

In the above inequalities the Hölder, 2D-Ladyžhenskaya, and Young inequalities have been applied, respectively. The estimates (3.16)-(3.17) lead to

$$\frac{d}{dt}\|\nabla \mathbf{u}_\alpha\|^2 + \nu \|\Delta \mathbf{u}_\alpha\|^2 \leq \frac{2}{\nu}\|\mathbf{f}\|^2 + \frac{C}{\nu}\|\overline{\mathbf{u}}_\alpha\| \|\nabla \overline{\mathbf{u}}_\alpha\|^2 \|\Delta \overline{\mathbf{u}}_\alpha\|.$$

and by using (3.15) we get

$$\begin{aligned}\|\nabla \mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 \, dt &\leq \frac{2\mathcal{F}}{\nu} + \frac{C}{\nu}\int_0^s \|\overline{\mathbf{u}}_\alpha\| \|\nabla \overline{\mathbf{u}}_\alpha\|^2 \|\Delta \overline{\mathbf{u}}_\alpha\| \, dt \\ &\leq \frac{2\mathcal{F}}{\nu} + \frac{C C_{SB}}{\nu \lambda_1} \int_0^s \|\Delta \overline{\mathbf{u}}_\alpha\|^2 \, dt \\ &\leq \frac{2\mathcal{F}}{\nu} + \frac{C C_{SB}^2}{\nu^2 \lambda_1} \qquad \forall\, s \geq 0.\end{aligned}$$

Therefore, the proof is complete. ∎

3.5 Estimates for the Modified Leray-α model

In this section we prove a uniform estimate for weak solutions to the modified Leray-α model

Lemma 3.4 (**ML-α**)**.** *Let $\mathbf{u}_0 \in \mathbf{V}$ and let $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$. Then, the unique weak solution $\mathbf{u}_\alpha$ of the ML-α model satisfies*

$$\|\nabla \mathbf{u}_\alpha(t)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 \, dt \leq \frac{C_{ML4}}{\nu^4} + \frac{2\mathcal{F}}{\nu} =: C_{ML_a} \qquad \forall\, s \geq 0, \tag{3.18}$$

where $C_{ML4} = C\, C_{ML1} C_{ML2} C_{ML3}$ with C is a positive constant, while for $i = 1, 2, 3$, the constants C_{MLi} are given by (3.21), (3.25) and (3.30), respectively.

Proof. The nonlinear term of this model is given by

$$N(\mathbf{u}_\alpha) = (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha.$$

This case requires more care than the previous ones, since the cancellations are less favorable. We prove it into 3 steps, starting with a $L^2 H^1_x \cap L^\infty_t L^2_x$ estimate of $\overline{\mathbf{u}}_\alpha$, then a $L^2 H^1_x \cap L^\infty_t L^2_x$ estimate of $\mathbf{u}_\alpha$, to finally get the conclusion.

Step 1. $L^2 H^1_x \cap L^\infty_t L^2_x$ estimate of $\overline{\mathbf{u}}_\alpha$. Taking $\overline{\mathbf{u}}_\alpha$ as test function in (2.8) and replacing $\mathbf{u}_\alpha$ by $\overline{\mathbf{u}}_\alpha - \alpha^2 \Delta \overline{\mathbf{u}}_\alpha$ we obtain

$$\frac{d}{dt}\left(\|\overline{\mathbf{u}}_\alpha\|^2 + \alpha^2 \|\nabla \overline{\mathbf{u}}_\alpha\|^2\right) + \nu \|\nabla \overline{\mathbf{u}}_\alpha\|^2 + 2\nu\alpha^2 \|\Delta \overline{\mathbf{u}}_\alpha\|^2 \leq \frac{1}{\nu\lambda_1}\|\mathbf{f}\|^2. \tag{3.19}$$

Here the fact $(P_\sigma[(\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], \overline{\mathbf{u}}_\alpha) = 0$ and the Poincaré inequality have been used on the r.h.s. Then one gets from (3.19)

$$\|\overline{\mathbf{u}}_\alpha(s)\|^2 + \alpha^2 \|\nabla \overline{\mathbf{u}}_\alpha(s)\|^2 + \nu \int_0^s \left(\|\nabla \overline{\mathbf{u}}_\alpha\|^2 + 2\alpha^2 \|\Delta \overline{\mathbf{u}}_\alpha\|^2\right) ds \leq C_{ML1} \qquad \forall s \geq 0, \tag{3.20}$$

where as in (3.11) above C_{ML1} is given by

$$\|\overline{\mathbf{u}}_0\|^2 + \alpha^2 \|\nabla \overline{\mathbf{u}}_0\|^2 + \frac{\mathcal{F}^2}{\nu\lambda_1} \leq (1 + \lambda_1)\|\mathbf{u}_0\|^2 + \frac{\mathcal{F}^2}{\nu\lambda_1} =: C_{ML1}. \tag{3.21}$$

Step 2. $L^2 H^1_x \cap L^\infty_t L^2_x$ estimate of $\mathbf{u}_\alpha$. Taking $\mathbf{u}_\alpha$ as test function in (2.8) yields

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_\alpha\|^2 + \nu \|\nabla \mathbf{u}_\alpha\|^2 = -((\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, \mathbf{u}_\alpha) + (\mathbf{f}, \mathbf{u}_\alpha). \tag{3.22}$$

The nonlinear term on the r.h.s of (3.22) can be now estimated by

$$\begin{aligned}
((\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, \mathbf{u}_\alpha) &\leq C\|\mathbf{u}_\alpha\|_4^2 \|\nabla \overline{\mathbf{u}}_\alpha\| \\
&\leq C\|\mathbf{u}_\alpha\| \|\nabla \mathbf{u}_\alpha\| \|\nabla \overline{\mathbf{u}}_\alpha\| \\
&\leq \frac{C}{\nu}\|\mathbf{u}_\alpha\|^2 \|\nabla \overline{\mathbf{u}}_\alpha\|^2 + \frac{\nu}{4}\|\nabla \mathbf{u}_\alpha\|^2.
\end{aligned}$$

Here we used the Hölder, 2D-Ladyžhenskaya, and Young inequalities, respectively. Using the Young inequality for the other term on the r.h.s of (3.22) gives

$$\frac{d}{dt}\|\mathbf{u}_\alpha\|^2 + \nu \|\nabla \mathbf{u}_\alpha\|^2 \leq \frac{2}{\lambda_1 \nu}\|\mathbf{f}\|^2 + \frac{C}{\nu}\|\mathbf{u}_\alpha\|^2 \|\nabla \overline{\mathbf{u}}_\alpha\|^2. \tag{3.23}$$

Using the estimate (3.20) leads to

$$\int_0^s \|\mathbf{u}_\alpha\|^2 \|\nabla \overline{\mathbf{u}}_\alpha\|^2 \, dt = \int_0^s \left(\|\overline{\mathbf{u}}_\alpha\|^2 + 2\alpha^2 \|\nabla \overline{\mathbf{u}}_\alpha\|^2 + \alpha^4 \|\Delta \overline{\mathbf{u}}_\alpha\|^2\right) \|\nabla \overline{\mathbf{u}}_\alpha\|^2 \, dt$$
$$\le \frac{4C_{ML1}^2}{\nu} \qquad \forall\, s \ge 0. \tag{3.24}$$

Here, we also used the following identity

$$\|\mathbf{u}_\alpha\|^2 = \|\overline{\mathbf{u}}_\alpha\|^2 + 2\alpha^2 \|\nabla \overline{\mathbf{u}}_\alpha\|^2 + \alpha^4 \|\Delta \overline{\mathbf{u}}_\alpha\|^2.$$

Therefore, by (3.23)-(3.24) we get

$$\|\mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{u}_\alpha\|^2 \, dt \le \frac{2\mathcal{F}}{\nu \lambda_1} + \frac{4CC_{ML1}^2}{\nu^2} =: C_{ML2} \qquad \forall\, s \ge 0. \tag{3.25}$$

Step 3. $L^2 H_x^2 \cap L_t^\infty H_x^1$ estimate of $\mathbf{u}_\alpha$. We take $-\Delta \mathbf{u}_\alpha$ as test function in (2.8) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}_\alpha\|^2 + \nu \|\Delta \mathbf{u}_\alpha\|^2 = (P_\sigma[(\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], \Delta \mathbf{u}_\alpha) - (\mathbf{f}, \Delta \mathbf{u}_\alpha). \tag{3.26}$$

The nonlinear term can be estimated as follows

$$(P_\sigma[(\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], \Delta \mathbf{u}_\alpha) \le C \|\mathbf{u}_\alpha\|_4 \|\nabla \overline{\mathbf{u}}_\alpha\|_4 \|\Delta \mathbf{u}_\alpha\|_2$$
$$\le C \|\mathbf{u}_\alpha\|^{1/2} \|\nabla \mathbf{u}_\alpha\|^{1/2} \|\nabla \overline{\mathbf{u}}_\alpha\|^{1/2} \|\Delta \mathbf{u}_\alpha\|^{3/2}$$
$$\le \frac{C}{\nu^3} \|\mathbf{u}_\alpha\|^2 \|\nabla \mathbf{u}_\alpha\|^2 \|\nabla \overline{\mathbf{u}}_\alpha\|^2 + \frac{\nu}{4} \|\Delta \mathbf{u}_\alpha\|^2, \tag{3.27}$$

by using the Hölder, 2D-Ladyžhenskaya, Sobolev, and Young inequalities, respectively. From (3.26)-(3.27) we obtain:

$$\frac{d}{dt}\|\nabla \mathbf{u}_\alpha\|^2 + \nu \|\Delta \mathbf{u}_\alpha\|^2 \le \frac{C}{\nu^3} \|\mathbf{u}_\alpha\|^2 \|\nabla \overline{\mathbf{u}}_\alpha\|^2 \|\nabla \mathbf{u}_\alpha\|^2 + \frac{2}{\nu}\|\mathbf{f}\|^2, \tag{3.28}$$

and in particular

$$\frac{d}{dt}\|\nabla \mathbf{u}_\alpha\|^2 \le \frac{CC_{ML2}}{\nu^3} \|\nabla \overline{\mathbf{u}}_\alpha\|^2 \|\nabla \mathbf{u}_\alpha\|^2 + \frac{2}{\nu}\|\mathbf{f}\|^2. \tag{3.29}$$

Hence, by (3.29) we obtain

$$\|\nabla \mathbf{u}_\alpha(s)\|^2 \le \left(\|\nabla \mathbf{u}_0\|^2 + \frac{2\mathcal{F}}{\nu}\right) \exp\left\{\frac{CC_{ML2}^2}{\nu^4}\right\} =: C_{ML3} \qquad \forall\, s \ge 0. \tag{3.30}$$

Together with (3.28) and (3.30) one obtains (3.18). Thus, the proof is complete also for this model. $\square$

4 The rate of convergence of $\mathbf{u}_\alpha$ to $\mathbf{u}$

In this section, we study the rate of convergence –in terms of α– of the weak solutions $\mathbf{u}_\alpha$ of the three α-models to the weak solution $\mathbf{u}$ of the NSE (in some suitable norms) as α tends to zero. We recall that, throughout this section the vector $\mathbf{e}$, defined as in (1.9), denotes the error between $\mathbf{u}$ and $\mathbf{u}_\alpha$ which are the weak solutions of the NSE (2.7) and of one of the α-models (2.8), respectively.

4.1 Error estimate in $L^2H^1_x \cap L^\infty_t L^2_x$

The first main result in this section is given by the following theorem:

Theorem 4.1. *Let* $\mathbf{u}_0 \in \mathbf{V}$ *and* $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$. *Then*

$$\|\mathbf{e}(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{e}\|^2 \, dt \le C_r \alpha^3 \qquad \forall\, s \ge 0, \tag{4.1}$$

where C_r *is given explicitly in:*

$$\begin{cases} (4.9) & \text{for the } L\text{-}\alpha \text{ model,} \\ (4.11) & \text{for the SB model,} \\ (4.12) & \text{for the } ML\text{-}\alpha \text{ model,} \end{cases}$$

Proof. As the three models share some common features, in a first step we consider these common ones, and in a second step we treat them separately to prove some specific estimates.

Step 1. Common features. We subtract (2.8) from (2.7) and by multiplying $\mathbf{e}$ and integrating by parts we get

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{e}\|^2 + \nu\|\nabla \mathbf{e}\|^2 = (-P_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] + P_\sigma[N(\mathbf{u}_\alpha)], \mathbf{e}). \tag{4.2}$$

We add and subtract on the r.h.s of (4.2) the term $((\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha, \mathbf{e})$ and then rewrite it in the following form:

$$\begin{aligned} RHS &= (-P_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] + P_\sigma[N(\mathbf{u}_\alpha)], \mathbf{e}) \\ &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + N(\mathbf{u}_\alpha), P_\sigma\, \mathbf{e}) \\ &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + N(\mathbf{u}_\alpha), \mathbf{e}) \\ &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e}) + (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + N(\mathbf{u}_\alpha), \mathbf{e}) =: I_1 + I_2, \end{aligned} \tag{4.3}$$

We will deal with the two terms (a common one and a residual term) on the r.h.s of (4.3) separately. Replacing $\mathbf{u}_\alpha$ by $\mathbf{u} - \mathbf{e}$, the first term in (4.3) is rewritten as follows:

$$\begin{aligned} I_1 = (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e}) &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)(\mathbf{u}-\mathbf{e}), \mathbf{e}) \\ &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}, \mathbf{e}) \\ &= ((-\mathbf{e}\cdot\nabla)\mathbf{u}, \mathbf{e}) \\ &= ((\mathbf{e}\cdot\nabla)\mathbf{e}, \mathbf{u}), \end{aligned}$$

where $(\mathbf{u}_\alpha\cdot\nabla)\mathbf{e}, \mathbf{e}) = 0$ has been used and the result is then estimated by

$$\begin{aligned} I_1 = ((\mathbf{e}\cdot\nabla)\mathbf{e}, \mathbf{u}) &\le C\|\mathbf{e}\|_4\|\nabla\mathbf{e}\|\|\mathbf{u}\|_4 \\ &\le C\|\mathbf{e}\|^{1/2}\|\nabla\mathbf{e}\|^{3/2}\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2} \\ &\le \frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\|\mathbf{e}\|^2 + \frac{\nu}{4}\|\nabla\mathbf{e}\|^2. \end{aligned} \tag{4.4}$$

The first inequality from above is due to the Hölder inequality with the pairing (1/4, 1/2, 1/4), the second one is obtained by applying the 2D-Ladyženskaya inequality and the last one comes from using the Young inequality with the pairing (1/4, 3/4).

The residual term $I_2 = (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + N(\mathbf{u}_\alpha), \mathbf{e})$ will be estimated for each model separately.

Step 2. Analysis specific for the various models

L-α model. For this model the nonlinear term is given by $N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$. The residual term is written as follows

$$I_2 = (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha, \mathbf{e}) = -(((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha) \cdot \nabla)\mathbf{u}_\alpha, \mathbf{e}).$$

The Hölder, 2D-Ladyžhenskaya, (1.8), (2.5), Sobolev, Poincaré, and Young inequalities are then used to get the following estimates:

$$\begin{aligned}
I_2 &\leq C\|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\|_4 \|\nabla \mathbf{u}_\alpha\| \|\mathbf{e}\|_4 \\
&\leq C\|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\|^{1/2} \|\nabla \mathbf{u}_\alpha - \nabla \overline{\mathbf{u}}_\alpha\|^{1/2} \|\nabla \mathbf{u}_\alpha\| \, \|\mathbf{e}\|^{1/2} \|\nabla \mathbf{e}\|^{1/2} \\
&\leq \frac{CC_L^{1/2}}{\lambda_1^{1/2}} \alpha^{3/2} \|\Delta \overline{\mathbf{u}}_\alpha\|^{1/2} \, \|\Delta \mathbf{u}_\alpha\|^{1/2} \, \|\nabla \mathbf{e}\| \\
&\leq \frac{CC_L^{1/2}}{\lambda_1^{1/2}} \alpha^{3/2} \|\Delta \mathbf{u}_\alpha\| \, \|\nabla \mathbf{e}\| \\
&\leq \frac{CC_L \alpha^3}{\nu \lambda_1} \|\Delta \mathbf{u}_\alpha\|^2 + \frac{\nu}{4}\|\nabla \mathbf{e}\|^2. \qquad (4.5)
\end{aligned}$$

Notice that $\|\nabla \mathbf{u}_\alpha(t)\|$ in the above estimate is uniformly bounded by $C_L^{1/2}$ where C_L given by Lemma 3.2. Collecting estimates (4.4) and (4.5) we obtain

$$\frac{d}{dt}\|\mathbf{e}\|^2 + \nu\|\nabla \mathbf{e}\|^2 \leq \frac{CC_L\alpha^3}{\nu\lambda_1}\|\Delta \mathbf{u}_\alpha\|^2 + \frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla \mathbf{u}\|^2\|\mathbf{e}\|^2. \qquad (4.6)$$

We are now going to apply the Gronwall's lemma for (4.6). Although the argument is standard we still provide the details for this model, while for the other ones the details will be skipped. Let us define

$$A(s) := -\frac{C}{\nu^3}\int_0^s \|\mathbf{u}\|^2\|\nabla \mathbf{u}\|^2 \, dt \qquad \forall\, s \geq 0,$$

where C is given in (4.6). Multiplying both sides of (4.6) by $A(t)$ yields

$$\|\mathbf{e}(s)\|^2 \leq \frac{CC_L\alpha^3}{\nu\lambda_1} \exp\{-A(s)\} \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 \, dt \qquad \forall\, s \geq 0, \qquad (4.7)$$

where we has used the facts that $A(s) \leq 0$ and $\mathbf{e}_0 = \mathbf{0}$. Thus, let us combine (4.7) with Lemmas 3.1 and 3.2 to prove uniform bounds for the modulus of $A(s)$ and for the integral from the r.h.s. to obtain

$$\|\mathbf{e}(s)\|^2 \leq \frac{CC_L^2\alpha^3}{\nu^2\lambda_1} \exp\left\{\frac{C_{NSE1}^2}{\nu^4}\right\} =: E_L\alpha^3 \qquad \forall\, s \geq 0, \qquad (4.8)$$

where C_L and C_{NSE1} are given by Lemmas 3.2 and 3.1, respectively. Finally, we combine (4.6) and (4.8) to get (4.1), with C_r given by

$$C_{r_L} = C\left(\frac{C_L}{\nu^2} + \frac{C_{NSE1}^2 E_L}{\nu^4}\right). \qquad (4.9)$$

SB model. In this case the residual term is given by

$$
\begin{aligned}
I_2 &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, \mathbf{e}) \\
&= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha - (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, \mathbf{e}) \\
&= -(((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha) \cdot \nabla)\mathbf{u}_\alpha, \mathbf{e}) - ((\overline{\mathbf{u}}_\alpha \cdot \nabla)(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha), \mathbf{e}) \\
&= R_1 + R_2. \qquad (4.10)
\end{aligned}
$$

The term R_1 on the r.h.s of (4.10) can be handled as (4.5) in the L-α model. The second term R_2 can be estimated as in the ML-α below, observing that $\|\overline{\mathbf{u}}\| \leq \|\mathbf{u}\|$. Therefore, the constant C_r in this case has the following form

$$C_{r_{SB}} = C_{r_L} + C_{r_{ML}}. \qquad (4.11)$$

ML-α model. In this case the residual term is rewritten as

$$
\begin{aligned}
R &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, \mathbf{e}) \\
&= ((\mathbf{u}_\alpha \cdot \nabla)\mathbf{e}, \mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha),
\end{aligned}
$$

and is handled precisely as in the L-α case. Then, the proof for this case follows by that of the L-α model, with C_r given by

$$C_{r_{ML_a}} := C\left(\frac{C_{ML_a}}{\nu^2} + \frac{C^2_{NSE1}E_{ML_a}}{\nu^4}\right), \qquad (4.12)$$

where C_{ML_a} is given by Lemma 3.4 and

$$E_{ML_a} := \frac{CC^2_{ML_a}}{\nu^2\lambda_1}\exp\left\{\frac{C^2_{NSE1}}{\nu^4}\right\}.$$

□

From Theorem 4.1 we have immediately the following results:

Corollary 4.1. *Let $\mathbf{u}_0 \in \mathbf{V}$ and let $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$. Then, it follows*

$$\|\overline{\mathbf{e}}(s)\|^2 + \nu \int_0^s \|\nabla\overline{\mathbf{e}}\|^2\, dt \leq 3C_r\alpha^3 \qquad \forall\, s \geq 0,$$

where $\overline{\mathbf{e}} = \overline{\mathbf{u}} - \overline{\mathbf{u}}_\alpha$ and C_r is given by Theorem 4.1 for each α-model.

Corollary 4.2. *Let $\mathbf{u}_0 \in \mathbf{V}$ and let $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$. Then, it follows*

$$\|(\mathbf{u} - \overline{\mathbf{u}}_\alpha)(s)\|^2 + \nu \int_0^s \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\|^2\, dt \leq C_{cor}(\alpha^2 + \alpha^3) \qquad \forall\, s \geq 0, \qquad (4.13)$$

where C_{cor} is given by (4.16).

Proof. The triangle inequality, Theorem 4.1, Lemma 3.1, relation (2.4), and Poincaré inequality yield for all $s \geq 0$

$$
\begin{aligned}
\|\mathbf{u} - \overline{\mathbf{u}}_\alpha(s)\|^2 &\leq 2\left(\|(\mathbf{u} - \mathbf{u}_\alpha)(s)\|^2 + \|(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)(s)\|^2\right) \\
&\leq 2C_r\alpha^3 + 2\alpha^4\|\Delta\overline{\mathbf{u}}_\alpha(s)\|^2 \\
&\leq 2C_r\alpha^3 + 2C\alpha^2\|\mathbf{u}_\alpha(s)\|^2 \\
&\leq 2C_r\alpha^3 + 2C\frac{C_E}{\lambda_1}\alpha^2. \qquad (4.14)
\end{aligned}
$$

Here, for each α-model C_E is given by C_L, C_{SB} or C_{ML_a} in Lemmas 3.2, 3.3 and 3.4, respectively. Moreover, C_r is given by Theorem 4.1. Similarly, we have

$$
\begin{aligned}
\nu\int_0^s \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\|^2\, dt &\leq 2\nu\left(\int_0^s \|\nabla(\mathbf{u} - \mathbf{u}_\alpha)\|^2\, dt + \int_0^s \|\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\|^2\, dt\right) \\
&\leq 2C_r\alpha^3 + 2C\alpha^2\nu\int_0^s \|\Delta\mathbf{u}_\alpha\|^2\, dt \\
&\leq 2C_r\alpha^3 + 2CC_E\alpha^2 \qquad \forall\, s \geq 0. \qquad (4.15)
\end{aligned}
$$

Thus, (4.13) follows by (4.14) and (4.15) with the constant C given by

$$
C_{cor} = 2\max\{C_r, CC_E, CC_E/\lambda_1\}. \qquad (4.16)
$$

□

4.2 Error estimate in $L^2H_x^2 \cap L_t^\infty H_x^1$

We now prove convergence rates in stronger norms, at the price of weaker rates. Throughout the rest of the paper, we assume $\alpha < L/2\pi$. Before going on to state the results, we start with a technical result, see [6, Prop. 4.2], that follows form a well-known result due to Brézis and Gallouët [4].

Lemma 4.1. *Let $0 \leq \alpha < \lambda_1^{-1/2} = L/2\pi$, and let $\mathbf{u}_\alpha$ be the weak solutions of any of α-models considered here. Then, there exist K_1 and K_2 such that*

$$
\|\overline{\mathbf{u}}_\alpha(t)\|_\infty^2 \leq K_1\log\left(\frac{L}{2\pi\alpha}\right) + K_2 \qquad \forall\, t \geq 0. \qquad (4.17)
$$

Proof. For the proof apply the same argument as in [6, Prop. 4.2], with the only difference that here due to the global estimates (we derived previously) we can work on arbitrary time intervals. □

We are now in order to state the next main result in this section.

Theorem 4.2. *Let $\alpha < L/2\pi$, $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$, and let us define*

$$
D(s) := \|\nabla\mathbf{e}(s)\|^2 + \nu\int_0^s \|\Delta\mathbf{e}\|^2\, dt \qquad \forall\, s \geq 0.
$$

Then, the following estimates hold true:
1. For the L-α model

$$
D(s) \leq C_R\,\alpha^2,
$$

where C_R is given by (4.27).
2. For the SB model

$$D(s) \le C_R\,\alpha^2\,(K_1 \log(L/2\pi\alpha) + K_2 + C_{SB})\,,$$

where C_R given by (4.29).
3. For the ML-α model

$$D(s) \le C_R\,\alpha^2\,(K_1 \log(L/2\pi\alpha) + K_2 + C_{ML_a})\,,$$

where C_R given by (4.34).
Here, the constants C_{ML_a}, C_{SB}, K_1, and K_2 are given by Lemmas 3.4, 3.3 and 4.1, respectively.

Proof. As before we first prove estimates valid for all models and then we pass to consider the specific ones. Subtracting (2.8) from (2.7) and taking $-\Delta\mathbf{e}$ as a test function yields:

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{e}\|^2 + \nu\|\Delta\mathbf{e}\|^2 = (-P_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] + P_\sigma[N(\mathbf{u}_\alpha)], -\Delta\mathbf{e}). \tag{4.18}$$

Adding and subtracting the term $((\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e})$ to the r.h.s of (4.18):

$$RHS = (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e}) + (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + N(\mathbf{u}_\alpha), -\Delta\mathbf{e}). \tag{4.19}$$

By recalling the definition $\mathbf{e} = \mathbf{u} - \mathbf{u}_\alpha$, the first term on the r.h.s of (4.19) can be split as follows:

$$\begin{aligned} I_1 &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e}) \\ &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + ((\mathbf{u}-\mathbf{e})\cdot\nabla)(\mathbf{u}-\mathbf{e}), -\Delta\mathbf{e}) \\ &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{e} - (\mathbf{e}\cdot\nabla)\mathbf{u} + (\mathbf{e}\cdot\nabla)\mathbf{e}, -\Delta\mathbf{e}) \\ &= ((\mathbf{u}\cdot\nabla)\mathbf{e}, \Delta\mathbf{e}) + (\mathbf{e}\cdot\nabla)\mathbf{u}, \Delta\mathbf{e}) =: I_{11} + I_{12}, \end{aligned} \tag{4.20}$$

where the vanishing of the term $((\mathbf{e}\cdot\nabla)\mathbf{e}, -\Delta\mathbf{e})$ has been used. The first term on the r.h.s of (4.20) is bounded by

$$\begin{aligned} I_{11} = ((\mathbf{u}\cdot\nabla)\mathbf{e}, \Delta\mathbf{e}) &\le C\|\mathbf{u}\|_4\|\nabla\mathbf{e}\|_4\|\Delta\mathbf{e}\| \\ &\le C\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2}\|\nabla\mathbf{e}\|^{1/2}\|\Delta\mathbf{e}\|^{3/2} \\ &\le \frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\|\nabla\mathbf{e}\|^2 + \frac{\nu}{6}\|\Delta\mathbf{e}\|^2. \end{aligned} \tag{4.21}$$

In (4.21), the Hölder, 2D-Ladyženskaya, and Young inequalities have been applied. Similarly, the other term on the r.h.s of (4.20) can be handled as follows:

$$\begin{aligned} I_{12} = ((\mathbf{e}\cdot\nabla)\mathbf{u}, \Delta\mathbf{e}) &\le C\|\mathbf{e}\|_4\|\nabla\mathbf{u}\|_4\|\Delta\mathbf{e}\| \\ &\le C\|\mathbf{e}\|^{1/2}\|\nabla\mathbf{e}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2}\|\Delta\mathbf{u}\|^{1/2}\|\Delta\mathbf{e}\| \\ &\le \frac{C}{\lambda_1^{1/2}}\|\nabla\mathbf{e}\|\|\nabla\mathbf{u}\|^{1/2}\|\Delta\mathbf{u}\|^{1/2}\|\Delta\mathbf{e}\| \\ &\le \frac{C}{\nu\lambda_1}\|\nabla\mathbf{e}\|^2\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\| + \frac{\nu}{6}\|\Delta\mathbf{e}\|^2. \end{aligned} \tag{4.22}$$

Using (4.21)-(4.22) the quantity I_1 in (4.20) can be bounded by

$$I_1 \le \left(\frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2 + \frac{C}{\nu\lambda_1}\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|\right)\|\nabla\mathbf{e}\|^2 + \frac{\nu}{3}\|\Delta\mathbf{e}\|^2. \tag{4.23}$$

In the following, we will estimate the second term I_2 from the r.h.s of (4.19), separately for each α-model.

L-α model. The nonlinear term is given in this case by $N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$. Therefore, the residual term I_2 can be estimated as follows

$$
\begin{aligned}
I_2 &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e}) \\
&= ((\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha) \cdot \nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e}) \\
&\leq C\|\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha\|_4 \|\nabla\mathbf{u}_\alpha\|_4 \|\Delta\mathbf{e}\| \\
&\leq C\|\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha\|^{1/2} \|\nabla\overline{\mathbf{u}}_\alpha - \nabla\mathbf{u}_\alpha\|^{1/2} \|\nabla\mathbf{u}_\alpha\|^{1/2} \|\Delta\mathbf{u}_\alpha\|^{1/2} \|\Delta\mathbf{e}\| \\
&\leq C\alpha \|\nabla\mathbf{u}_\alpha\| \|\Delta\mathbf{u}_\alpha\| \|\Delta\mathbf{e}\| \\
&\leq \frac{C}{\nu}\alpha^2 \|\nabla\mathbf{u}_\alpha\|^2 \|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{6}\|\Delta\mathbf{e}\|^2 \\
&\leq \frac{CC_L\alpha^2}{\nu}\|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{6}\|\Delta\mathbf{e}\|^2, \qquad (4.24)
\end{aligned}
$$

where the Hölder, 2D-Ladyžhenskaya, (1.8)-(2.4), and Young inequalities have been applied. Moreover, C_L is given by Lemma 3.2. Using estimates (4.18)-(4.24) leads to

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\mathbf{e}\|^2 + \nu\|\Delta\mathbf{e}\|^2 \leq \left(\frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2 + \frac{C}{\nu\lambda_1}\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\| \right) \|\nabla\mathbf{e}\|^2 \\
+ \frac{CC_L\alpha^2}{\nu}\|\Delta\mathbf{u}_\alpha\|^2, \qquad (4.25)
\end{aligned}
$$

and we can rewrite it as follows

$$
y'(t) - g(t)y(t) \leq h(t) \qquad \forall\, t \geq 0,
$$

where for all $t \geq 0$

$$
\begin{cases}
y(t) := \|\nabla\mathbf{e}(t)\|^2, \\
g(t) := \dfrac{C}{\nu^3}\|\mathbf{u}(t)\|^2\|\nabla\mathbf{u}(t)\|^2 + \dfrac{C}{\nu\lambda_1}\|\nabla\mathbf{u}(t)\|\|\Delta\mathbf{u}(t)\|, \\
h(t) := \dfrac{CC_L\alpha^2}{\nu}\|\Delta\mathbf{u}_\alpha(t)\|^2.
\end{cases}
$$

Therefore, since $\nabla\mathbf{e}(0) = \mathbf{0}$, an application of the Gronwall's lemma gives

$$
\|\nabla\mathbf{e}(s)\|^2 \leq \frac{CC_L}{\nu^2} \exp\left\{ \frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2}C_{NSE2}^{1/2}}{\nu^2\lambda_1} \right\} \alpha^2 =: R_L\alpha^2 \qquad \forall\, s \geq 0. \qquad (4.26)
$$

Finally, combining (4.25) and (4.26) yields

$$
\|\nabla\mathbf{e}(s)\|^2 + \nu\int_0^s \|\Delta\mathbf{e}\|^2\, dt \leq C_R\,\alpha^2 \qquad \forall\, s \geq 0,
$$

where

$$
C_R = \left(\frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2}C_{NSE2}^{1/2}}{\nu^2\lambda_1} \right) R_L + \frac{CC_L^2}{\nu^2}. \qquad (4.27)
$$

SB model. In this case the nonlinear term is given by $N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha$ and adding and subtracting the term $(\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$ lead us to

$$
\begin{aligned}
I_2 &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, -\Delta\mathbf{e}) \\
&= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha - (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, -\Delta\mathbf{e}) \\
&=: I_{21} + I_{22}. \qquad (4.28)
\end{aligned}
$$

Here, the first term on the r.h.s of (4.28) can be handled as follows

$$
\begin{aligned}
I_{21} &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e}) \\
&= ((\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha) \cdot \nabla)\mathbf{u}_\alpha, \Delta\mathbf{e}),
\end{aligned}
$$

which is similar to (4.24) in the L-α model. The other term can be rewritten as follows

$$
\begin{aligned}
I_{22} &= (-(\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, -\Delta\mathbf{e}) \\
&= ((\overline{\mathbf{u}}_\alpha \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}),
\end{aligned}
$$

which turns out to be similar to (4.32) in the ML-α model. Therefore, the constant C_R in this case is similar as in the ML-α model and has the form

$$
C_R = \left(\frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2} C_{NSE2}^{1/2}}{\nu^2 \lambda_1} \right) R_{SB} + \frac{C C_{SB}}{\nu^2}. \qquad (4.29)
$$

Here C_{SB} is given by Lemma 3.3 and

$$
R_{SB} := \frac{C C_{SB}}{\nu^2} \exp\left\{ \frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2} C_{NSE2}^{1/2}}{\nu^2 \lambda_1} \right\}.
$$

Thus, the proof is complete.

ML-α model. The nonlinear term is given now by $N(\mathbf{u}_\alpha) = (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha$ and the residual term can be rewritten as follows

$$
\begin{aligned}
I_2 &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, -\Delta\mathbf{e}) \\
&= ((\mathbf{u}_\alpha \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}) \\
&= (((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha) \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}) + ((\overline{\mathbf{u}}_\alpha \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}) \\
&=: I_{21} + I_{22}. \qquad (4.30)
\end{aligned}
$$

The first term from the r.h.s of (4.30) can be estimated by

$$
\begin{aligned}
I_{21} &= (((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha) \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}) \\
&\le C\|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\|_4 \|\nabla\overline{\mathbf{u}}_\alpha - \nabla\mathbf{u}_\alpha\|_4 \|\Delta\mathbf{e}\|_2 \\
&\le C\|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\|^{1/2} \|\nabla\overline{\mathbf{u}}_\alpha - \nabla\mathbf{u}_\alpha\| \, \|\Delta\overline{\mathbf{u}}_\alpha - \Delta\mathbf{u}_\alpha\|^{1/2} \|\Delta\mathbf{e}\| \\
&\le C\alpha\|\Delta\mathbf{u}_\alpha\| \|\nabla\mathbf{u}_\alpha\| \|\Delta\mathbf{e}\| \\
&\le \frac{C}{\nu}\alpha^2 \|\nabla\mathbf{u}_\alpha\|^2 \|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{12}\|\Delta\mathbf{e}\|^2 \\
&\le \frac{C C_{ML_\alpha}}{\nu}\alpha^2 \|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{12}\|\Delta\mathbf{e}\|^2, \qquad (4.31)
\end{aligned}
$$

where C_{ML_a} is given by Lemma 3.4. Next, we bound the second term on the r.h.s of (4.30) as follows (recall in the all section we are in the case $\alpha < L/2\pi$):

$$\begin{aligned}
I_{22} &= ((\overline{\mathbf{u}}_\alpha \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta \mathbf{e}) \\
&\le C\|\overline{\mathbf{u}}_\alpha\|_\infty \|\nabla \overline{\mathbf{u}}_\alpha - \nabla \mathbf{u}_\alpha\| \|\Delta \mathbf{e}\| \\
&\le C\alpha \|\overline{\mathbf{u}}_\alpha\|_\infty \|\Delta \mathbf{u}_\alpha\| \|\Delta \mathbf{e}\| \\
&\le \frac{C\alpha^2}{\nu} \|\overline{\mathbf{u}}_\alpha\|_\infty^2 \|\Delta \mathbf{u}_\alpha\|^2 + \frac{\nu}{12}\|\Delta \mathbf{e}\|^2 \\
&\le \frac{C\alpha^2}{\nu}\left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2\right)\|\Delta \mathbf{u}_\alpha\|^2 + \frac{\nu}{12}\|\Delta \mathbf{e}\|^2,
\end{aligned} \tag{4.32}$$

Here, K_1 and K_2 are given in Lemma 4.1. Therefore,

$$I_{22} \le \frac{CK_2\alpha^2}{\nu}\|\Delta \mathbf{u}_\alpha\|^2 + \frac{\nu}{12}\|\Delta \mathbf{e}\|^2 \quad \text{ for } \alpha < \frac{L}{2\pi}.$$

Here in (4.31)-(4.32), we have used the inequalities Hölder, 2D-Ladyzhenshaya, Young and formula (4.17) in Lemma 4.1. Putting (4.23) and (4.31)-(4.32) into the r.h.s of (4.18), we obtain

$$\begin{aligned}
\frac{d}{dt}\|\nabla \mathbf{e}\|^2 + \nu\|\Delta \mathbf{e}\|^2 \le & \left(\frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla \mathbf{u}\|^2 + \frac{C}{\nu\lambda_1}\|\nabla \mathbf{u}\|\|\Delta \mathbf{u}\|\right)\|\nabla \mathbf{e}\|^2 \\
& + \frac{C\alpha^2}{\nu}\left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_{ML_a}\right)\|\Delta \mathbf{u}_\alpha\|^2,
\end{aligned} \tag{4.33}$$

Since inequality (4.33) shares a similar structure with (4.25) then the rest of the proof follows by that of the L-α model. The constant C_R in this case is given by

$$C_R = \left(\frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2}C_{NSE2}^{1/2}}{\nu^2\lambda_1}\right)R_{ML_a} + \frac{CC_{ML_a}}{\nu^2}. \tag{4.34}$$

Here

$$R_{ML_a} := \frac{CC_{ML_a}}{\nu^2}\exp\left\{\frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2}C_{NSE2}^{1/2}}{\nu^2\lambda_1}\right\}.$$

Thus, the proof is complete. □

From the Theorem 4.2 we can easily deduce the following corollaries for related errors.

Corollary 4.3. *Let* $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$, *and let us define*

$$E(s) := \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)(s)\|^2 + \nu\int_0^s \|\Delta(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\|^2\, dt \qquad \forall\, s \ge 0.$$

Then, it follows

$$E(s) \le 2C_R\, h(\alpha) + 2CC_E\alpha^2 \qquad \forall\, s \ge 0, \tag{4.35}$$

where

$$h(\alpha) = \begin{cases} \alpha^2 & \textit{for the L-}\alpha\textit{ model}, \\ \alpha^2\left(K_1\log(L/2\pi\alpha) + K_2 + C_{SB}\right) & \textit{for the SB model}, \\ \alpha^2\left(K_1\log(L/2\pi\alpha) + K_2 + C_{ML_a}\right) & \textit{for the ML-}\alpha\textit{ model}. \end{cases}$$

Proof. The proof shares the same idea with Corollary 4.2. We start with

$$\begin{aligned}\|\nabla(\mathbf{u}-\bar{\mathbf{u}}_\alpha)(s)\|^2 &\leq 2(\|\nabla(\mathbf{u}-\mathbf{u}_\alpha)(s)\|^2 + \|\nabla(\mathbf{u}_\alpha-\bar{\mathbf{u}}_\alpha)(s)\|^2) \\ &\leq 2C_R\, h(\alpha) + 2\alpha^4\|\nabla\Delta\bar{\mathbf{u}}_\alpha(s)\|^2 \\ &\leq 2C_R\, h(\alpha) + 2C\alpha^2\|\nabla\mathbf{u}_\alpha(s)\|^2 \\ &\leq 2C_R\, h(\alpha) + 2CC_E\alpha^2 \qquad \forall\, s \geq 0, \end{aligned} \tag{4.36}$$

where (2.4) has been used in the third inequality. The constant C_E is defined as in Corollary 4.2. Similarly, for all $s \geq 0$

$$\begin{aligned} I &= \nu\int_0^s \|\Delta\mathbf{u} - \Delta\bar{\mathbf{u}}_\alpha\|^2\, dt \\ &\leq 2\nu\int_0^s \|\Delta\mathbf{u} - \Delta\mathbf{u}_\alpha\|^2\, dt + 2\nu\int_0^s \|\Delta\mathbf{u}_\alpha - \Delta\bar{\mathbf{u}}_\alpha\|^2\, dt \\ &\leq 2C_R h(\alpha) + 2\nu\alpha^4\int_0^s \|\Delta\Delta\bar{\mathbf{u}}_\alpha\|^2\, dt \\ &\leq 2C_R h(\alpha) + 2\alpha^2\nu\int_0^s \|\Delta\mathbf{u}_\alpha\|^2\, dt \\ &\leq 2C_R h(\alpha) + 2CC_E\alpha^2. \end{aligned} \tag{4.37}$$

Therefore, (4.35) follows by combining (4.36) and (4.37). □

5 The rate of convergence of p_α to p

In this section we focus on the order of the error of the pressure, by using the results from the previous sections. Let p and p_α be the pressures associated to the weak solutions $\mathbf{u}$ and $\mathbf{u}_\alpha$ of the NSE (1.1)-(1.3) and all α-models (1.4)-(1.6), respectively. It will be shown that the difference

$$q := p - p_\alpha$$

is bounded in terms of the parameter α, uniformly in time, in a suitable norm.

Theorem 5.1. *Let $\mathbf{u}_0 \in \mathbf{V}$, let $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$, and let us define*

$$I(s) := \int_0^s \|\nabla q\|^2\, dt \qquad \forall\, s \geq 0.$$

Then, the following estimates hold true:
1. For for the L-α model

$$I(s) \leq C\alpha^{5/2} + C\alpha^3 \qquad \forall\, s \geq 0,$$

where C given by (5.5).

2. For for the SB model

$$I(s) \leq C\alpha^3 + C\alpha^{5/2}(\log(L/2\pi\alpha)+1)^{1/2} + C\alpha^2(\log(L/2\pi\alpha)+1) \qquad \forall\, s \geq 0,$$

where C is given by (5.9).

3. For for the ML-α model

$$I(s) \leq C\alpha^4 + C\alpha^3 + C(\alpha^{5/2}+\alpha^2)(\log(L/2\pi\alpha)+1) \qquad \forall\, s \geq 0,$$

where C given by (5.14).

Proof. It follows subtracting from the NSE (1.1)-(1.3) the α-model (1.4)-(1.6) that

$$-\Delta q = \nabla \cdot [(\mathbf{u} \cdot \nabla)\mathbf{u} - N(\mathbf{u}_\alpha)] =: \nabla \cdot \mathbf{g}. \tag{5.1}$$

We are assuming that p and p_α are periodic and with zero average. The vanishing of the mean values of p and p_α ensure their uniqueness (up to an arbitrary function of time). Multiplying (5.1) by q and integrating on $\mathbb{T}_2$ the Cauchy-Schwarz inequality yields

$$\|\nabla q\|^2 \leq \|\mathbf{g}\|^2 = \int_{\mathbb{T}_2} |(\mathbf{u} \cdot \nabla)\mathbf{u} - N(\mathbf{u}_\alpha)|^2 \, d\mathbf{x}. \tag{5.2}$$

In order to estimate the error of the pressure we are led to bound the r.h.s of (5.2). Replacing $\mathbf{e}$ by $\mathbf{u} - \mathbf{u}_\alpha$, adding and subtracting the term $(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha$ gives

$$\begin{aligned}
\|\mathbf{g}\|^2 &= \int_{\mathbb{T}_2} |(\mathbf{u} \cdot \nabla)\mathbf{u} - N(\mathbf{u}_\alpha)|^2 \, d\mathbf{x} \\
&= \int_{\mathbb{T}_2} |(\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha - N(\mathbf{u}_\alpha)|^2 \, d\mathbf{x} \\
&= \int_{\mathbb{T}_2} |-(\mathbf{e} \cdot \nabla)\mathbf{u} + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{e} + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha - N(\mathbf{u}_\alpha)|^2 \, d\mathbf{x} \\
&\leq C \int_{\mathbb{T}_2} \left(|(\mathbf{e} \cdot \nabla)\mathbf{u}|^2 + |(\mathbf{u}_\alpha \cdot \nabla)\mathbf{e}|^2 + |(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha - N(\mathbf{u}_\alpha)|^2\right) d\mathbf{x}.
\end{aligned} \tag{5.3}$$

By (5.3) one has for all $s \geq 0$:

$$I(s) = \int_0^s \|\mathbf{g}\|^2 \, dt \leq C(I_1 + I_2 + I_3). \tag{5.4}$$

The estimate is given for each α-model separately.

L-α model. In this case we have $N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$. Each term on the r.h.s of (5.4) will be estimated below. First,

$$\begin{aligned}
I_1 &= \int_0^s \int_{\mathbb{T}_2} |(\mathbf{e} \cdot \nabla)\mathbf{u}|^2 \, d\mathbf{x} \, dt \\
&\leq \int_0^s \|\mathbf{e}\|_4^2 \|\nabla \mathbf{u}\|_4^2 \, dt \\
&\leq \int_0^s \|\mathbf{e}\| \, \|\nabla \mathbf{e}\| \, \|\nabla \mathbf{u}\| \, \|\Delta \mathbf{u}\| \, dt \\
&\leq C_r^{1/2} C_{NSE2}^{1/2} \alpha^{3/2} \left(\int_0^s \|\nabla \mathbf{e}\|^2 \, dt\right)^{1/2} \left(\int_0^s \|\Delta \mathbf{u}\|^2 \, dt\right)^{1/2} \\
&\leq \frac{C_r C_{NSE}}{\nu} \alpha^3 \qquad \forall \, s \geq 0,
\end{aligned}$$

where we have used the Hölder and 2D-Ladyženskaya inequalities, Lemma 3.1, 4.1, and 4.2. Next, we have

$$\begin{aligned}
I_2 &= \int_0^s \int_{\mathbb{T}_2} |(\mathbf{u}_\alpha \cdot \nabla)\mathbf{e}|^2 \, d\mathbf{x} \, dt \\
&\leq \int_0^s \|\mathbf{u}_\alpha\| \, \|\nabla \mathbf{u}_\alpha\| \, \|\nabla \mathbf{e}\| \, \|\Delta \mathbf{e}\| \, dt \\
&\leq \frac{C_L}{\lambda^{1/2}} \left(\int_0^s \|\nabla \mathbf{e}\|^2 \, dt\right)^{1/2} \left(\int_0^s \|\Delta \mathbf{e}\|^2 \, dt\right)^{1/2} \\
&\leq \frac{C_L C_r^{1/2} C_R^{1/2}}{\nu} \alpha^{5/2} \qquad \forall \, s \geq 0,
\end{aligned}$$

here we have used the Hölder and 2D-Ladyžhenskaya inequalities, Lemma 3.2, Theorems 4.1 and 4.2, respectively. Finally

$$
\begin{aligned}
I_3 &= \int_0^s \int_{\mathbb{T}_2} |((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\cdot\nabla)\mathbf{u}_\alpha|^2 \, d\mathbf{x}\, dt \\
&\le \int_0^s \|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\| \, \|\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\| \, \|\nabla \mathbf{u}_\alpha\| \, \|\Delta \mathbf{u}_\alpha\| \, dt \\
&\le 2CC_L\alpha^3 \int_0^s \|\Delta \overline{\mathbf{u}}_\alpha\| \|\Delta \mathbf{u}_\alpha\| \, dt \\
&\le \frac{2CC_L^{3/2}}{\nu}\alpha^3 \qquad \forall\, s \ge 0,
\end{aligned}
$$

here in addition we have used (1.8), (4.36), and Lemma 3.2. Thus the proof of the convergence rate for this model follows by collecting the previous estimates

$$
I(s) \le \frac{C_L C_r^{1/2} C_R^{1/2}}{\nu}\alpha^{5/2} + \left(\frac{C_r C_{NSE}}{\nu} + \frac{2CC_L^{3/2}}{\nu}\right)\alpha^3 \qquad \forall\, s \ge 0. \tag{5.5}
$$

SB model. For this model we have for all $s \ge 0$

$$
\begin{aligned}
I(t) &= \int_0^s \int_{\mathbb{T}_2} |(\mathbf{u}\cdot\nabla)\mathbf{u} - (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha - (\overline{\mathbf{u}}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha|^2 \, d\mathbf{x}dt \\
&\le 4(I_1 + I_2 + I_3). \qquad (5.6)
\end{aligned}
$$

One has used the fact that $\mathbf{u}_\alpha = \mathbf{u} - \mathbf{e}$ in the second term inside the integral. Similarly, by using Corollary 4.1 and Lemma 3.3 we get

$$
\begin{aligned}
I_1 &= \int_0^s \int_{\mathbb{T}_2} |((\mathbf{u} - \overline{\mathbf{u}}_\alpha)\cdot\nabla)\mathbf{u}|^2 \, d\mathbf{x}dt \\
&\le \int_0^s \|\mathbf{u} - \overline{\mathbf{u}}_\alpha\| \, \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\| \, \|\nabla \mathbf{u}\| \, \|\Delta \mathbf{u}\| \\
&\le \frac{C_{NSE2}^{1/2}}{\nu} C_{cor}^{1/2}(\alpha^2 + \alpha^3)^{1/2} \left(\nu \int_0^s \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\|^2 \, dt\right)^{1/2} \left(\nu \int_0^s \|\Delta \mathbf{u}\|^2 \, dt\right)^{1/2} \\
&\le \frac{C_{NSE2}}{\nu} C_{cor}(\alpha^2 + \alpha^3) \qquad \forall\, s \ge 0. \qquad (5.7)
\end{aligned}
$$

We deal with the second integral by using Lemma 3.3 and Theorem 4.1-4.2 to show

$$
\begin{aligned}
I_2 &= \int_0^s \int_{\mathbb{T}_2} |(\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{e}|^2 \, d\mathbf{x}dt \\
&\le \int_0^s \|\overline{\mathbf{u}}_\alpha\| \|\nabla \overline{\mathbf{u}}_\alpha\| \|\nabla \mathbf{e}\| \|\Delta \mathbf{e}\| \, d\mathbf{x}dt \\
&\le \frac{CC_{SB}}{\nu\lambda_1^{1/2}} \left(\nu \int_0^s \|\nabla \mathbf{e}\|^2 \, dt\right)^{1/2} \left(\nu \int_0^s \|\Delta \mathbf{e}\|^2 \, dt\right)^{1/2} \\
&\le \frac{CC_{SB}}{\nu\lambda_1^{1/2}} C_r^{1/2} C_R^{1/2} \alpha^{5/2} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_{SB}\right)^{1/2} \qquad \text{since } \alpha < \frac{L}{2\pi}, \qquad (5.8)
\end{aligned}
$$

Similarly, the last term can be estimated for all $s \geq 0$ by

$$
\begin{aligned}
I_3 &= \int_0^s \int_{\mathbb{T}_2} |(\overline{\mathbf{u}}_\alpha \cdot \nabla)(\mathbf{u} - \overline{\mathbf{u}}_\alpha)|^2 \, d\mathbf{x} dt \\
&\leq \int_0^s \|\overline{\mathbf{u}}_\alpha\| \|\nabla \overline{\mathbf{u}}_\alpha\| \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\| \|\Delta(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\| \, dt \\
&\leq \frac{C_{SB}^{1/2}}{\nu \lambda_1^{1/2}} (2C_R \, h(\alpha) + 2CC_{SB}\alpha^2)^{1/2} \left(\nu \int_0^s \|\nabla \mathbf{u}_\alpha\|^2 \, dt \right)^{1/2} \left(\nu \int_0^s \|\Delta(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\|^2 \, dt \right)^{1/2} \\
&\leq \frac{C_{SB}}{\nu \lambda_1^{1/2}} (2C_R \, h(\alpha) + 2CC_{SB}\alpha^2) \\
&= \frac{CC_{SB}}{\nu \lambda_1^{1/2}} \left[C_R \, \alpha^2 \left(K_1 \log\left(\frac{L}{2\pi\alpha} \right) + K_2 + C_{SB} \right) + C_{SB}\alpha^2 \right] \qquad \text{since } \alpha < \frac{L}{2\pi}.
\end{aligned}
$$

Therefore, by the above estimates we get

$$
\begin{aligned}
& I(s) \leq \\
& \leq \frac{C_{NSE2}}{\nu} C_{cor}(\alpha^2 + \alpha^3) + \frac{CC_{SB}}{\nu \lambda_1^{1/2}} C_r^{1/2} C_R^{1/2} \alpha^{5/2} \left(K_1 \log\left(\frac{L}{2\pi\alpha} \right) + K_2 + C_{SB} \right)^{1/2} \\
& \quad + \frac{CC_{SB}}{\nu \lambda_1^{1/2}} \left[C_R \left(K_1 \log\left(\frac{L}{2\pi\alpha} \right) + K_2 + C_{SB} \right) + C_{SB} \right] \alpha^2.
\end{aligned} \tag{5.9}
$$

Thus the proof for this model is completed.

ML-α model. For this model I_1 is estimated as above. We start with I_2 by

$$
\begin{aligned}
I_2 &\leq \int_0^s \|\mathbf{u}_\alpha\| \, \|\nabla \mathbf{u}_\alpha\| \, \|\nabla \mathbf{e}\| \, \|\Delta \mathbf{e}\| \, dt \\
&\leq \frac{C_{ML_a}}{\lambda_1^{1/2}} \left(\int_0^s \|\nabla \mathbf{e}\|^2 \, dt \right)^{1/2} \left(\int_0^s \|\Delta \mathbf{e}\|^2 \, dt \right)^{1/2} \\
&\leq \frac{C_{ML_a} C_r^{1/2} C_R^{1/2}}{\nu} \alpha^{5/2} \left(K_1 \log\left(\frac{L}{2\pi\alpha} \right) + K_2 + C_{ML_a} \right)^{1/2},
\end{aligned} \tag{5.10}
$$

for all $s \geq 0$. One has used the results Lemma 3.4, Theorems 4.1 and 4.2. The term I_3 is bounded by

$$
I_3 = \int_0^s \int_{\mathbb{T}_2} |(\mathbf{u}_\alpha \cdot \nabla)(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)|^2 \, d\mathbf{x} \, dt \leq 2(I_{31} + I_{32}). \tag{5.11}
$$

By (4.36) and Lemma 3.4 yield

$$
\begin{aligned}
I_{31} &= \int_0^s \int_{\mathbb{T}_2} |((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha) \cdot \nabla)(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)|^2 \, d\mathbf{x} \, dt \\
&\leq \int_0^t \|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\|_4^2 \, \|\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\|_4^2 \, dt \\
&\leq \int_0^t \|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\| \, \|\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\|^2 \|\Delta(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\| \, dt \\
&\leq \frac{CC_{ML_a}^2}{\nu} \alpha^4 \qquad \forall \, s \geq 0.
\end{aligned} \tag{5.12}
$$

The other term can be estimated for all $s \geq 0$ by

$$
\begin{aligned}
I_{32} &= \int_0^s \int_{\mathbb{T}_2} |(\overline{\mathbf{u}}_\alpha \cdot \nabla)(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)|^2 \, d\mathbf{x} \, dt \\
&\leq \int_0^s \|\overline{\mathbf{u}}_\alpha\|_\infty^2 \|\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\|^2 \, dt \\
&\leq \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 \right) \alpha^2 \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 \, dt \\
&\leq \frac{C_{ML_a}}{\nu} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 \right) \alpha^2, \qquad (5.13)
\end{aligned}
$$

here Lemma 4.1 and (2.5) have been applied. There for the proof for this model is finished by (5.10)-(5.13)

$$
\begin{aligned}
I(s) \leq \frac{C_r C_{NSE}}{\nu} \alpha^3 &+ \frac{C_{ML_a} C_r^{1/2} C_R^{1/2}}{\nu} \alpha^{5/2} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_{ML_a} \right)^{1/2} \\
&+ \frac{C C_{ML_a}^2}{\nu} \alpha^4 + \frac{C_{ML_a}}{\nu} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 \right) \alpha^2, \qquad (5.14)
\end{aligned}
$$

which concludes the proof. □

6 The 3D case: a few additional remarks

The problem in $\mathbb{R}^3$ is rather different since the solution of the NSE are not known to be globally smooth and the available estimates for the convective term are different from those employed in the previous sections. More specifically to the problems studied here, the rate of convergence has been estimated in the 3D case by Chen, Guenther, Kim, Thomann, and Waymire [9]. They proved the following estimate.

$$
\int_0^T \|\mathbf{e}\| \, dt \leq C(T)\alpha.
$$

Their analysis is carried out in the 3D periodic setting and assumes a small data condition (such that existence and uniqueness of weak solutions $\mathbf{u}$ of the 3D NSE is ensured). Here $\mathbf{u}$ and $\mathbf{u}_\alpha$ are the weak solutions of the NSE and Navier-Stokes-α, respectively, with periodic boundary conditions.

Another result concerning the convergence rate for the same α-models of turbulence considered in the previous sections has been obtained in [14]. For both for 2D and 3D it is proved that

$$
\sup_{t \in [0,T]} \|\mathbf{e}(t)\|^2 + \int_0^T \|\nabla \mathbf{e}\|^2 \, dt \leq C(T)\alpha^2.
$$

The result is obtained with $\mathbf{u}_\alpha(0,\cdot) = \mathbf{u}_0 \in \mathbf{V}, \mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{H})$, and under an extra assumption that the weak solution of the 3D NSE is such that $\mathbf{u} \in \mathbf{L}^4(0,T;\mathbf{H}^1(\mathbb{T}_3))$. The latter condition ensures existence and uniqueness of weak solutions. The logarithmic term in (1.10) is removed in the results for both 2D and 3D periodic cases in [14].

This section is devoted to find out the convergence rate of weak solutions of the α-models to that of the NSE in the 3D case. If $\mathbf{u} \in \mathbf{L}^4([0,T];\mathbf{H}^1(\mathbb{T}_3))$, the standard Sobolev embedding implies that $\mathbf{u} \in \mathbf{L}^4([0,T];\mathbf{L}^6(\mathbb{T}_2))$ which is a special case of the well-known Leray-Serrin-Prodi (LSP) 3D uniqueness assumption, where $r = 4$ and $s = 6$, see formula (6.1) below, see Leray [28], Prodi [31] and Serrin [33]. More specifically, that is

$$\mathbf{u} \in \mathbf{L}^r([0,T];\mathbf{L}^s(\mathbb{T}_2)) \quad \text{where} \quad \frac{3}{s} + \frac{2}{r} = 1,\ s \geq 3. \tag{6.1}$$

It is also known, see for example Galdi [18, Definition 2.1, Theorem 4.2], that weak solutions which satisfy the LSP condition are unique and regular in the set of all Leray-Hopf weak solutions. Recently, under the conditions $\mathbf{f} \in \mathbf{L}^2([0,T];\mathbf{H})$, $\mathbf{u}_0 \in \mathbf{H}^1(\mathbb{T}_2)$ and an extra condition $\mathbf{u} \in \mathbf{L}^4([0,T];\mathbf{H}^1(\mathbb{T}_3))$, the author of [14] showed that the rate of convergence of weak solutions $\mathbf{u}_\alpha$ of the three α-models to $\mathbf{u}$ is $\mathcal{O}(\alpha)$ for some suitable norms. More precisely, that is

$$\sup_{t\in[0,T]} \|\mathbf{e}(t)\|^2 + \nu \int_0^T \|\nabla \mathbf{e}\|^2\, dt \leq C(T)\alpha^2,$$

where C is the Sobolev constant and C_T is given by

$$C(T) = C_1 \exp\left\{ \frac{C}{\nu^3} \int_0^T \|\nabla \mathbf{u}\|^4\, ds \right\}. \tag{6.2}$$

here $C_1 = C_1(\mathbf{u}_0, \mathbf{f}, \nu)$. On one hand, it follows that in the case $\mathbf{u} \in \mathbf{L}^4(\mathbb{R}_+;\mathbf{H}^1(\mathbb{T}_2))$, which satisfies (6.1), we get the error is uniformly bounded in time, i.e.,

$$\sup_{t\geq 0} \|\mathbf{e}(t)\|^2 + \nu \int_0^\infty \|\nabla \mathbf{e}\|^2\, dt \leq C_\infty \alpha^2,$$

where

$$C_\infty = C_1 \exp\left\{ \frac{C}{\nu^3} \int_0^\infty \|\nabla \mathbf{u}\|^4\, ds \right\}.$$

On the other hand, if a weak solution $\mathbf{u}$ of the NSE regular up to a time $T_* < \infty$ and cannot be smoothly extended, we say that $\mathbf{u}$ becomes irregular at the time T_* (or that T_* is an epoch of irregularity). Assume that T_* is the first time that $\mathbf{u}$ becomes irregular, see Galdi [18, Def. 6.1], then it is well-known that the $\mathbf{H}^1(\mathbb{T}_3)$-norm of $\mathbf{u}$, $\|\nabla \mathbf{u}(t)\|^2$ will blow-up as t approaches T_* from below, see for instance [18, Theorem 6.4], Leray [28] and Scheffer [32]. More specifically, there exists $\epsilon = \epsilon_{T_*} > 0$ small enough such that

$$\|\nabla \mathbf{u}(t)\| \geq \frac{C\nu^{3/4}}{(T_* - t)^{1/4}} \qquad \forall\, t \in (T_* - \epsilon, T_*), \tag{6.3}$$

where $C > 0$ is only depending on $\mathbb{T}_2$. In that case, by (6.3), we consider $C(T)$ in (6.2) with $T_* - \epsilon < T < T_*$, which will also blow-up as in the following way

$$\begin{aligned}
C(T) &= C_1 \exp\left\{ \frac{C}{\nu^3} \int_0^T \|\nabla \mathbf{u}\|^4\, ds \right\} \\
&\geq C_1 \exp\left\{ \frac{C}{\nu^3} \int_{T_*-\epsilon}^T \|\nabla \mathbf{u}\|^4\, ds \right\} \\
&\geq C_1 \exp\left\{ C \int_{T_*-\epsilon}^T \frac{1}{T_* - s}\, ds \right\} \\
&= C_1 \frac{\epsilon^C}{(T_* - T)^C},
\end{aligned}$$

showing the effect of being T_* an epoch of irregularity on the convergence rate.

7 Conclusions

In this work, after assuming the not so restrictive assumptions on the data $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$, we provided the rate of convergence, as $\alpha \to 0^+$, of $\mathbf{u}_\alpha$ to $\mathbf{u}$ as well as of p_α to p. In addition our argument is tied up to the periodic case mostly because of special properties of the Stokes operator A and of the convective term in this setting. The extension of the results to other boundary conditions such as the Dirichlet boundary conditions or to the Euler equations are left as future works. In the 3D case extra-assumptions for the uniqueness of solution of the NSE are probably necessary to be assumed, to obtain rates of convergence.

Remark 7.1. *It seems to be the case that all results herein can be established when the periodic domain $\mathbb{T}_2 = \mathbb{R}^2/[0, L]^2$ is replaced by the whole space $\mathbb{R}^2$, following the approach developed in [30]. However, the existence and uniqueness of weak solutions of all α-models herein needs to be studied carefully. Also this issue will be investigated in a forthcoming work.*

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